NOTE ON HOW I GOT THE COMPARISON FUNCTION FOR THE FIRST EXAMPLE FROM CLASS

I decided to write a note on how I got the comparison function for the first example in which we used the comparison test. Hopefully this will help you with those kinds of problems on the homework.

We had to decide whether $\int_1^\infty \frac{1}{\sqrt{1+x^3}}$ was convergent or divergent. The way you can deduce this from a comparison test is either

- by bounding the function $\frac{1}{\sqrt{1+x^3}}$ below by a function whose integral from 1 to ∞ is divergent OR
- by bounding the function $\frac{1}{\sqrt{1+x^3}}$ above by a function whose integral from 1 to ∞ is convergent.

We know the behavior of $\int_1^\infty \frac{1}{x^p}$ for any positive p: it converges if $p > 1$ and it diverges if $p \leq 1$. So the comparison test would help IF we can either show

- $\frac{1}{x^p} \leq \frac{1}{\sqrt{1+x^3}}$ for all $x \geq 1$, for some $p \leq 1$ OR
- $\frac{1}{\sqrt{1+x^3}} \leq \frac{1}{x^p}$ for all $x \geq 1$, for some $p > 1$.

Of course these two options are exclusive.

For example, if we tried to compare with $\frac{1}{x^3}$, since $\int_1^\infty \frac{1}{x^3}$ converges, that would only be a useful comparison if $\frac{1}{\sqrt{1+x^3}} \leq \frac{1}{x^3}$ FOR ALL $x \geq 1$, which would be true if and only if $\sqrt{1+x^3} \geq x^3$ FOR ALL $x \geq 1$. This clearly fails for x big enough, for example, it fails for $x = 2$.

However, note that

$$x^3 + 1 > x^3, \text{ FOR ALL } x \geq 1,$$

therefore

$$\sqrt{1+x^3} > \sqrt{x^3} \text{ FOR ALL } x \geq 1.$$

Therefore

$$\frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{x^3}} \text{ FOR ALL } x \geq 1.$$

And now the integral from $\frac{1}{\sqrt{x^3}} = \frac{1}{x^{\frac{3}{2}}}$ is convergent because $\frac{3}{2} > 1$. So that tells us that our integral converges, too.

Let's see what other powers p we could have picked that would work. Since we showed $\frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{x^3}}$ for all $x \geq 1$, then for any p with the property that $\frac{1}{x^{\frac{3}{2}}} < \frac{1}{x^p}$ we would also have

$$\frac{1}{\sqrt{1+x^3}} < \frac{1}{x^p} \text{ for all } x \geq 1.$$

So let's see for what p we have

$$\frac{1}{x^{\frac{3}{2}}} \leq \frac{1}{x^p} \text{ for all } x \geq 1.$$

Well this is true precisely when

$$x^p \leq x^{\frac{3}{2}} \text{ for all } x \geq 1.$$

And now this is true precisely when

$$p \leq \frac{3}{2}.$$

Remark 1. Note that for the last implication I used that $x^a < x^b \iff a < b$ for which it is *crucial* that x is bigger than 1. If x were between 0 and 1, for example, the inequality would get reversed: $(\frac{1}{2})^3 < (\frac{1}{2})^2$ but $3 > 2$.

Ok, so for any $p \leq \frac{3}{2}$, we have that

$$\frac{1}{\sqrt{1+x^3}} < \frac{1}{x^p} \text{ for all } x \geq 1.$$

Now, any $p \leq 1$ does not help because for those the corresponding integral is divergent and having our function bounded above by a function with divergent integral doesn't help. So the only values of p that do help are the ones in between 1 (not including 1) and $\frac{3}{2}$. So you could pick $p = 1.234$ or $p = 1.45564$.

But since in order to deduce al of this we used our initial comparison with $\frac{1}{x^{\frac{3}{2}}}$ and that works and it was easy to prove that it is larger than the given function, we might as well use that.